# Math Logic: Model Theory & Computability
# Lecture 08

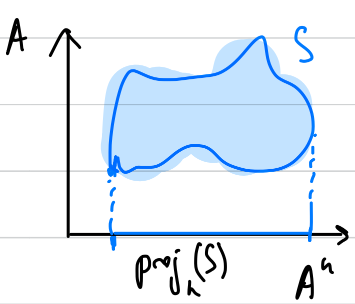
## Definability as a geometric/descriptive concept (without σ-formulas).

Let $\underline{A} := (A, \sigma)$ be a σ-structure. For a set $P \subseteq A$, we denote by $Def^n_{\underline{A}}(P)$ the set of all P-definable subsets of $A^n$ (relations of arity n).
Let $Def_{\underline{A}}(P) := \bigcup_{n \in \mathbb{N}} Def^n_{\underline{A}}(P)$.

**Def.** A family $\mathcal{B} \subseteq Pow(A)$ of subsets of A is called a (Boolean) algebra if $\emptyset \in \mathcal{B}$ and $\mathcal{B}$ is closed under complements ($S \in \mathcal{B} \Rightarrow A \setminus S \in \mathcal{B}$) and finite unions ($S_1, S_2 \in \mathcal{B} \Rightarrow S_1 \cup S_2 \in \mathcal{B}$).

**Obs1.** For each $n \in \mathbb{N}$, $Def^n_{\underline{A}}(P)$ is an algebra.

The connectives ¬ and ∨ correspond to the set operations complement and union. What set operation does ∃ correspond to?




$S = \{(\vec{a}, b) \in A^n \times A : \underline{A} \models \varphi(\vec{a}, b, \vec{p})\}$.

Then $\{\vec{a} \in A^n : \underline{A} \models \exists u\, \varphi(\vec{a}, u, \vec{p})\} =$
$= \{\vec{a} \in A^n : \text{there is } b \in A \text{ s.t. } \underline{A} \models \varphi(\vec{a}, b, \vec{p})\}$
$= \{\vec{a} \in A^n : \text{there is } b \in A \text{ s.t. } (\vec{a}, b) \in S\}$
$= proj_n(S)$.

**Obs2.** $Def_{\underline{A}}(P)$ is closed under projections (= images of projections).

Adding a dummy variable to the vector extending a formula corresponds to taking preimages under projections:

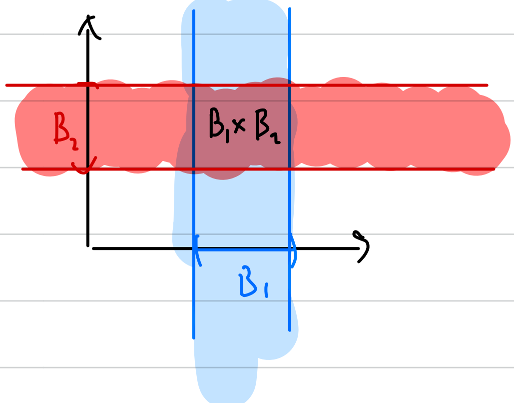
**Obs 3.** $\mathrm{Def}_{\underline{A}}(P)$ is closed under preimages of projections, i.e. if $B \in \mathrm{Def}^n_{\underline{A}}(P)$ then $B \times A \in \mathrm{Def}^{n+1}_{\underline{A}}(P)$.

We can also permute the order of variables in an extended formula, and this corresponds to permuting the coordinates of a definable set:

**Obs 4.** $\mathrm{Def}_{\underline{A}}(P)$ is closed under permutation of coordinates, i.e. for $n \in \mathbb{N}^f$ and a permutation $\pi$ of $\{1, 2, \dots, n\}$, we think of $\pi$ as $A^n \mapsto A^n$ by $(a_i)_{i \le n} \mapsto (a_{\pi(i)})_{i \le n}$, and if $B \in \mathrm{Def}^n_{\underline{A}}(P)$, then $\pi(B) \in \mathrm{Def}^n_{\underline{A}}(P)$.

**Cor.** $\mathrm{Def}_{\underline{A}}(P)$ is closed under Cartesian products.

**Proof.** Let $B_1 \in \mathrm{Def}^{n_1}_{\underline{A}}(P)$ and $B_2 \in \mathrm{Def}^{n_2}_{\underline{A}}(P)$, then by the closure under pre-projections, we have that $B_1 \times A^{n_2}, A^{n_1} \times B_2 \in \mathrm{Def}^{n_1+n_2}_{\underline{A}}(P)$, so because $\mathrm{Def}^{n_1+n_2}_{\underline{A}}(P)$ is an algebra, we get $B_1 \times B_2 = (B_1 \times A^{n_2}) \cap (A^{n_1} \times B_2) \in \mathrm{Def}^{n_1+n_2}_{\underline{A}}(P)$. $\square$

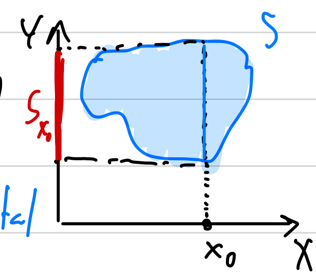



Lastly, plugging-in parametres from $P$ corresponds taking fibers over points in $P$.

**Def.** For sets $X, Y$, and $S \subseteq X \times Y$, and a point $x_0 \in X$, we call the set

$$S_{x_0} := \{ y \in Y : (x_0, y) \in S \}$$

the *vertical fiber of $S$ over $x_0$*.




Also, for $y_0 \in Y$, we call $S^{y_0} := \{ x \in X : (x, y_0) \in S \}$ the *horizontal fiber of $S$ over $y_0$*.

**Obs 5.** $\mathrm{Def}_{\underline{A}}(P)$ is closed under taking fibers over points in $P$, i.e. if $B \in \mathrm{Def}^{n+1}_{\underline{A}}(P)$ and $p \in P$, then $B_p \in \mathrm{Def}^n_{\underline{A}}(P)$.

This motivates the following definition.

**Def.** Let $A$ be a set and for each $n \in \mathbb{N}$, let $\mathcal{D}_n$ be a collection of subsets of $A^n$. For a set $P \subseteq A$, we call $\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ *$P$-constructive* if

(i) $\mathcal{D}_n$ is an algebra for each $n \in \mathbb{N}$.
(ii) $\mathcal{D}$ is closed under (images of) projections.
(iii) $\mathcal{D}$ is closed under preimages of projections.
(iv) $\mathcal{D}$ is closed under permutations of coordinates.
(v) $\mathcal{D}$ is closed under taking fibers over points in $P$.

Observations 1–5 imply that $\mathrm{Def}_{\underline{A}}(P)$ is $P$-constructive, and in fact:

**Theorem.** For a $\sigma$-structure $\underline{A} := (A, \sigma)$ and $P \in A$, the collection $\mathrm{Def}_{\underline{A}}(P)$ is the $P$-constructive collection generated by (the smallest $P$-constructive collection containing):
(i) constants: $\{c^{\underline{A}}\}$ for each $c \in \mathrm{Const}(\sigma)$,
(ii) graphs of functions: $G_{f^{\underline{A}}}$ for each $f \in \mathrm{Funct}(\sigma)$,
(iii) relations: $R^{\underline{A}}$ for each $R \in \mathrm{Rel}(\sigma)$.

*Proof.* We already know that $\mathrm{Def}_{\underline{A}}(P)$ is such a collection and to show that it's the smallest, take another such collection $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ and show that $\mathrm{Def}_{\underline{A}}(P) \subseteq \mathcal{D}$ by induction on the construction of formulas. This is left as an exercise. $\square$

**Theories, models, and axiomatizations.** For a signature $\sigma$, a *$\sigma$-theory* is just a set of $\sigma$-sentences. For a $\sigma$-theory $T$, we refer to the sentences in $T$ as the *axioms* of $T$.

**Def.** For a nonempty $\sigma$-structure $\underline{M}$ and a $\sigma$-theory $T$, we say that $\underline{M}$ satisfies (or models) $T$, denoted $\underline{M} \models T$, if $\underline{M} \models \varphi$ for all $\varphi \in T$. Equivalently, we call $\underline{M}$ a $\sigma$-model of $T$. We denote by $\mathcal{M}_\sigma(T)$ the class of all $\sigma$-models of $T$.

**Def.** For a collection $\mathcal{C}$ of $\sigma$-structures, we say that a $\sigma$-theory $T$ is an axiomatization for $\mathcal{C}$ if $\mathcal{C} = \mathcal{M}_\sigma(T)$. For $\sigma$-theories $T_1, T_2$, we say that $T_1$ and $T_2$ are equivalent if $\mathcal{M}_\sigma(T_1) = \mathcal{M}_\sigma(T_2)$.
A $\sigma$-theory $T$ is said to be finitely axiomatizable if there is a finite $\sigma$-theory $T'$ equivalent to $T$.

**Examples.** (a) Let $\sigma$ be a signature. For fixed $n \in \mathbb{N}^+$, the class $\mathcal{C}_{\leq n}$ of all $\sigma$-structures of cardinality $\leq n$ is axiomatized by the following sentence:

$$\varphi_{\leq n} := \exists x_1 \exists x_2 \ldots \exists x_n \forall y \, (y = x_1 \vee y = x_2 \vee \ldots \vee y = x_n).$$

Thus, the class $\mathcal{C}_{>n}$ of all $\sigma$-structures with $>n$ elements is axiomatized by $\varphi_{>n} := \neg \varphi_{\leq n}$.
Finally, the class $\mathcal{C}_\infty$ of all infinite $\sigma$-structures is axiomatized by the theory

$$T_\infty := \{\varphi_{>1}, \varphi_{>2}, \varphi_{>3}, \ldots\} = \{\varphi_{>n} : n \in \mathbb{N}^+\}.$$